

Parallel and sequential hybrid methods for a finite family of **asymptotically quasi ϕ -nonexpansive mappings**

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Abstract In this paper we study some novel parallel and sequential hybrid methods for finding a common fixed point of a finite family of asymptotically quasi ϕ -nonexpansive mappings. The results presented here modify and extend some previous results obtained by several authors.

Keywords Asymptotically quasi- ϕ -nonexpansive mapping · Common fixed point · Hybrid method · Parallel and sequential computation.

1 Introduction

Let C be a nonempty closed convex subset of a Banach space E . A mapping $T : C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

In 2005, Matsushita and Takahashi [21] proposed the following hybrid method, combining Mann iterations with projections onto closed convex subsets, for finding a fixed point of a relatively nonexpansive mapping T :

$$\begin{aligned} x_0 &\in C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ Q_n &= \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, n \geq 0. \end{aligned}$$

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This algorithm has been modified and generalized for finding a common fixed point of a finite or infinite family of relatively nonexpansive mappings by several authors, such as Takahashi et al. [29], Takahashi and Zembayashi [30], Wang and Xuan [32], Reich and Sabach [24, 25], Kang, Su, and Zhang [13], Plubtieng and Ungchittrakool [22], etc...

In 2011, Liu [20] introduced the following cyclic method for a finite family of relatively nonexpansive mappings:

$$\begin{aligned} x_0 &\in C, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_{n(\text{mod})N}x_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, n \geq 0. \end{aligned}$$

Very recently, Anh and Chung [3] considered the following parallel method for a finite family of relatively nonexpansive mappings:

$$\begin{aligned} x_0 &\in C, \\ y_n^i &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), \quad i = 1, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_n &= \{v \in C : \phi(v, \bar{y}_n) \leq \phi(v, x_n)\}, \\ Q_n &= \{v \in C : \langle Jx_0 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, n \geq 0. \end{aligned}$$

According to this algorithm, the intermediate approximations y_n^i can be found in parallel. Then among all $y_n^i, i = 1, \dots, N$, the farthest element from x_n , denoted by \bar{y}_n , is chosen. After that, two convex closed subsets C_n and Q_n containing the set of common fixed points are constructed. The next approximation x_{n+1} is defined as the generalized projection of x_0 onto the intersection $C_n \cap Q_n$.

Further, some generalized hybrid projection methods have been introduced for families of hemi-relatively or weak relatively nonexpansive mappings (see, [13, 27, 31]).

On the other hand, there has been an increasing interest in the class of **asymptotically quasi ϕ -nonexpansive mappings** (c.f., [5, 7, 9, 10, 11, 12, 14, 18, 19, 28, 33]), which is a generalization of the class of quasi ϕ -nonexpansive mappings. The last one contains the class of relatively nonexpansive mappings as a proper subclass.

Unfortunately, many hybrid algorithms for (relatively) nonexpansive mappings cannot be directly extended to **asymptotically quasi ϕ -nonexpansive mappings**.

The aim of this paper is to combine a parallel splitting-up technique proposed in [3] with a monotone hybrid iteration method (see, [26]) for finding a common fixed point of a finite family of **asymptotically quasi ϕ -nonexpansive mappings**. The organization of the paper is as follows: In Section 2 we collect some definitions and results which are used in this paper. Section 3 deals with the convergence analysis of the proposed parallel and sequential hybrid algorithms. **Finally, a numerical example shows that even in the sequential mode, our parallel hybrid method is faster than the corresponding sequential one [20].**

2 Preliminaries

In this section we recall some definitions and results needed for further investigation. We refer the interested reader to [2, 8] for more details.

Definition 1 A Banach space X is called

- 1) strictly convex if the unit sphere $S_1(0) = \{x \in X : \|x\| = 1\}$ is strictly convex, i.e., the inequality $\|x + y\| < 2$ holds for all $x, y \in S_1(0), x \neq y$;
- 2) uniformly convex if for any given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1, \|x - y\| = \varepsilon$ the inequality $\|x + y\| \leq 2(1 - \delta)$ holds;
- 3) smooth if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (1)$$

exists for all $x, y \in S_1(0)$;

- 4) uniformly smooth if the limit (1) exists uniformly for all $x, y \in S_1(0)$.

Let E be a real Banach space with the dual E^* and $J : E \rightarrow 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in E^* : \langle f, x \rangle = \|x\|^2 = \|f\|^2 \right\}.$$

The following basic properties of the geometry of E and its normalized duality mapping J can be found in [4]:

- i) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm to weak* continuous;
- ii) If E is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \rightarrow 2^{E^*}$ is single-valued, one-to-one, and onto;
- iii) If E is a uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E ;
- iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex;
- v) Each uniformly convex Banach space E has the Kadec-Klee property, i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$.

Next we assume that C is a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E . Consider the Lyapunov functional $\phi : E \times E \rightarrow R_+$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \forall x, y \in E.$$

From the definition of ϕ , we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \quad (2)$$

The generalized projection $\Pi_C : E \rightarrow C$ is defined by

$$\Pi_C(x) = \arg \min_{y \in C} \phi(x, y).$$

Lemma 1 [1] Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E . Then the following conclusions hold:

- i) $\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \leq \phi(x, y), \forall x \in C, y \in E$;
- ii) if $x \in E, z \in C$, then $z = \Pi_C(x)$ iff $\langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$;
- iii) $\phi(x, y) = 0$ iff $x = y$.

Lemma 2 [1] Let E be a uniformly convex and uniformly smooth real Banach space, $\{x_n\}$ and $\{y_n\}$ be two sequences in E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E , $T : C \rightarrow C$ be a mapping, and $F(T)$ be the set of fixed points of T . A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightarrow p$ and $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow +\infty$. The set of all asymptotic fixed points of T will be denoted by $\tilde{F}(T)$.

Definition 2 A mapping $T : C \rightarrow C$ is called

- i) relatively nonexpansive mapping if $F(T) \neq \emptyset, F(T) = \tilde{F}(T)$, and

$$\phi(p, Tx) \leq \phi(p, x), \forall p \in F(T), \forall x \in C;$$

- ii) closed if for any sequence $\{x_n\} \subset C, x_n \rightarrow x$ and $Tx_n \rightarrow y$, then $Tx = y$;
- iii) quasi ϕ - nonexpansive mapping (or hemi-relatively nonexpansive mapping) if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall p \in F(T), \forall x \in C;$$

- iv) **asymptotically quasi ϕ -nonexpansive** if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \rightarrow 1$ as $n \rightarrow +\infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \forall n \geq 1, \forall p \in F(T), \forall x \in C;$$

- v) uniformly L - Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \forall n \geq 1, \forall x, y \in C.$$

Lemma 3 [5] Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and C be a nonempty closed convex subset of E . Let $T : C \rightarrow C$ be a closed and **asymptotically quasi ϕ -nonexpansive** mapping with a sequence $\{k_n\} \subset [1, +\infty), k_n \rightarrow 1$. Then $F(T)$ is a closed convex subset of C .

Lemma 4 [5, 15, 21] Let E be a strictly convex reflexive smooth Banach space, A be a maximal monotone operator of E into E^* , and $J_r = (J + rA)^{-1}J : E \rightarrow D(A)$ be the resolvent of A with $r > 0$. Then,

- i) $F(J_r) = A^{-1}0$;
- ii) $\phi(u, J_r x) \leq \phi(u, x)$ for all $u \in A^{-1}0$ and $x \in E$.

Lemma 5 [26] Let E be a uniformly convex and uniformly smooth Banach space, A be a maximal monotone operator from E to E^* , and J_r be a resolvent of A . Then J_r is closed hemi-relatively nonexpansive mapping.

3 Main results

3.1 Parallel hybrid methods

Assume that $T_i, i = 1, 2, \dots, N$, are **asymptotically quasi ϕ -nonexpansive** mappings with a sequence $\{k_n^i\} \subset [1, +\infty), k_n^i \rightarrow 1$, i.e., $F(T_i) \neq \emptyset$, and

$$\phi(p, T_i^n x) \leq k_n^i \phi(p, x), \forall n \geq 1, \forall p \in F(T_i), \forall x \in C.$$

Throughout this paper we suppose that the set $F = \bigcap_{i=1}^N F(T_i)$ is nonempty.

Then, putting $k_n := \max\{k_n^i : i = 1, \dots, N\}$, we have $k_n \subset [1, +\infty), k_n \rightarrow 1$, and

$$\phi(p, T_i^n x) \leq k_n \phi(p, x), \forall i = 1, \dots, N, \forall n \geq 1, \forall p \in F, \forall x \in C.$$

In the following theorems we will assume that the set $F = \bigcap_{i=1}^N F(T_i)$ is nonempty and bounded in C , i.e., there exists a positive number ω such that $F \subset \Omega := \{u \in C : \|u\| \leq \omega\}$.

Theorem 1 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of **asymptotically quasi ϕ -nonexpansive** mappings with a sequence $\{k_n\} \subset [1, +\infty), k_n \rightarrow 1$. Moreover, suppose for each $i \geq 1$, the mapping T_i is uniformly L_i -Lipschitz continuous and the set $F = \bigcap_{i=1}^N F(T_i)$ is nonempty and bounded in C . Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in C, C_0 := C, \\ y_n^i &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i^n x_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &:= \{v \in C_n : \phi(v, \bar{y}_n) \leq \phi(v, x_n) + \varepsilon_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

where $\varepsilon_n := (k_n - 1)(\omega + \|x_n\|)^2$, and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_0$.

Proof The proof of Theorem 1 is divided into five steps.

Step 1. Claim that F and C_n are closed and convex subsets of C .

Indeed, from the uniform L_i -Lipschitz continuity of T_i , T_i is L_i -Lipschitz continuity. Hence T_i is continuous. This implies that T_i is closed. By Lemma 3, $F(T_i)$ is closed and convex subset of C for all $i = 1, 2, \dots, N$. Hence, $F = \bigcap_{i=1}^N F(T_i)$ is closed and convex. Further, $C_0 = C$ is closed and convex by the assumption. Suppose that C_n is a closed and convex subset of C for some $n \geq 0$. From the inequality $\phi(v, \bar{y}_n) \leq \phi(v, x_n) + \varepsilon_n$, we obtain

$$\langle v, Jx_n - J\bar{y}_n \rangle \leq \frac{1}{2} (\|x_n\|^2 - \|\bar{y}_n\|^2 + \varepsilon_n).$$

Therefore,

$$C_{n+1} = \left\{ v \in C_n : \langle v, Jx_n - J\bar{y}_n \rangle \leq \frac{1}{2} (\|x_n\|^2 - \|\bar{y}_n\|^2 + \varepsilon_n) \right\},$$

which implies that C_{n+1} is closed and convex. Thus, C_n is closed and convex subset of C for all $n \geq 0$, and $\Pi_C x_0$ and $\Pi_{C_n} x_0$ are well-defined.

Step 2. Claim that $F \subset C_n$ for all $n \geq 0$.

Observe first that $F \subset C_0 = C$. Now suppose $F \subset C_n$ for some $n \geq 0$. For each $u \in F$, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
 \phi(u, \bar{y}_n) &= \|u\|^2 - 2\langle u, J\bar{y}_n \rangle + \|\bar{y}_n\|^2 \\
 &= \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n) \langle u, JT_{i_n}^n x_n \rangle \\
 &\quad + \|\alpha_n Jx_n + (1 - \alpha_n) JT_{i_n}^n x_n\|^2 \\
 &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1 - \alpha_n) \langle u, JT_{i_n}^n x_n \rangle \\
 &\quad + \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|T_{i_n}^n x_n\|^2 \\
 &= \alpha_n \phi(u, x_n) + (1 - \alpha_n) \phi(u, T_{i_n}^n x_n) \\
 &\leq \alpha_n \phi(u, x_n) + k_n (1 - \alpha_n) \phi(u, x_n) \\
 &\leq \phi(u, x_n) + (k_n - 1)(1 - \alpha_n) \phi(u, x_n) \\
 &\leq \phi(u, x_n) + (k_n - 1)(\omega + \|x_n\|)^2 \\
 &= \phi(u, x_n) + \varepsilon_n.
 \end{aligned}$$

This implies that $u \in C_{n+1}$. Hence $F \subset C_{n+1}$. By induction, we obtain $F \subset C_n$ for all $n \geq 0$. For each $u \in F \subset C_n$, by $x_n = \Pi_{C_n} x_0$ and Lemma 1, we have

$$\phi(x_n, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0).$$

Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. The boundedness of the sequence $\{x_n\}$ is followed from relation (2).

Step 3. Claim that the sequence $\{x_n\}$ converges strongly to some point $p \in C$ as $n \rightarrow \infty$.

By the construction of C_n , we have $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$. Now taking into account $x_n = \Pi_{C_n} x_0$, $x_{n+1} \in C_n$ and using Lemma 1, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) - \phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0).$$

This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing. Therefore, the limit of $\{\phi(x_n, x_0)\}$ exists. We also have $x_m \in C_m \subset C_n$ for all $m \geq n$. From Lemma 1 and $x_n = \Pi_{C_n} x_0$, we obtain

$$\phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0) \rightarrow 0,$$

as $m, n \rightarrow \infty$. This together with Lemma 2 implies that $\|x_m - x_n\| \rightarrow 0$. Hence, $\{x_n\}$ is Cauchy sequence. Since E is complete and C is closed, we get

$$\lim_{n \rightarrow \infty} x_n = p \in C. \quad (3)$$

Step 4. Claim that $p \in F$.

Indeed, observing that

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \rightarrow 0, \quad (4)$$

and

$$\|x_{n+1} - x_n\| \rightarrow 0. \quad (5)$$

In view of $x_{n+1} \in C_{n+1}$ and by the construction of C_{n+1} , we obtain

$$\phi(x_{n+1}, \bar{y}_n) \leq \phi(x_{n+1}, x_n) + \varepsilon_n. \quad (6)$$

Recalling that the set F and the sequence $\{x_n\}$ are bounded, and putting $M = \sup \{\|x_n\| : n = 1, 2, \dots\}$, we get

$$\varepsilon_n = (k_n - 1)(\omega + \|x_n\|)^2 \leq (k_n - 1)(\omega + M)^2 \rightarrow 0. \quad (7)$$

From (4), (6), (7), we obtain $\phi(x_{n+1}, \bar{y}_n) \rightarrow 0$ as $n \rightarrow \infty$. This together with Lemma 2 implies that $\|x_{n+1} - \bar{y}_n\| \rightarrow 0$. Therefore, from (5), $\|x_n - \bar{y}_n\| \rightarrow 0$. Further, by the definition of i_n , we have $\|x_n - y_n^i\| \leq \|x_n - \bar{y}_n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $i = 1, 2, \dots, N$, hence, from (3) we obtain

$$\lim_{n \rightarrow \infty} y_n^i = p, \quad i = 1, 2, \dots, N. \quad (8)$$

From the relation $y_n^i = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i^n x_n)$ we obtain

$$\|Jy_n^i - JT_i^n x_n\| = \alpha_n \|Jx_n - JT_i^n x_n\|. \quad (9)$$

Observing that $\{x_n\}$ is bounded, T_i is uniformly L_i -Lipschitz continuous, and the solution set F is not empty, we have $\|Jx_n - JT_i^n x_n\| \leq \|Jx_n\| + \|JT_i^n x_n\| = \|x_n\| + \|T_i^n x_n\| \leq \|x_n\| + \|T_i^n x_n - T_i^n \xi\| + \|\xi\| \leq \|x_n\| + L_i \|x_n - \xi\| + \|\xi\|$, where $\xi \in F$ is an arbitrary fixed element. The last inequality proves the boundedness of the sequence $\{\|Jx_n - JT_i^n x_n\|\}$. Using $\lim_{n \rightarrow \infty} \alpha_n = 0$, from (9), we find

$$\lim_{n \rightarrow \infty} \|Jy_n^i - JT_i^n x_n\| = 0.$$

Since $J^{-1} : E^* \rightarrow E$ is uniformly continuous on each bounded subset of E^* , the last relation implies $\lim_{n \rightarrow \infty} \|y_n^i - T_i^n x_n\| = 0$. Hence, from (8) we obtain

$$\lim_{n \rightarrow \infty} T_i^n x_n = p, \quad i = 1, \dots, N. \quad (10)$$

By (3), (10) and the uniform L_i -Lipschitz continuity of T_i , we have

$$\begin{aligned} \|T_i^{n+1} x_n - T_i^n x_n\| &\leq \|T_i^{n+1} x_n - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| \\ &\leq (L_i + 1) \|x_{n+1} - x_n\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_n - T_i^n x_n\| \rightarrow 0. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} T_i^{n+1} x_n = p$, i.e., $T_i^{n+1} x_n = T_i T_i^n x_n \rightarrow p$ as $n \rightarrow \infty$. In view of the continuity of T_i and (10), it follows that $T_i p = p$ for all $i = 1, 2, \dots, N$. Therefore $p \in F$.

Step 5. Claim that $p = x^\dagger := \Pi_F(x_0)$.

Indeed, since $x^\dagger = \Pi_F(x_0) \in F \subset C_n$ and $x_n = \Pi_{C_n}(x_0)$, from Lemma 1, we have

$$\phi(x_n, x_0) \leq \phi(x^\dagger, x_0) - \phi(x^\dagger, x_n) \leq \phi(x^\dagger, x_0). \quad (11)$$

Therefore,

$$\begin{aligned}\phi(x^\dagger, x_0) &\geq \lim_{n \rightarrow \infty} \phi(x_n, x_0) = \lim_{n \rightarrow \infty} \left\{ \|x_n\|^2 - 2 \langle x_n, Jx_0 \rangle + \|x_0\|^2 \right\} \\ &= \|p\|^2 - 2 \langle p, Jx_0 \rangle + \|x_0\|^2 \\ &= \phi(p, x_0).\end{aligned}$$

From the definition of x^\dagger , it follows that $p = x^\dagger$. The proof of Theorem 1 is complete.

Remark 1 If in Theorem 1 instead of the uniform Lipschitz continuity of the operators T_i , $i = 1, \dots, N$, we require their closedness and asymptotical regularity [6], i.e., for any bounded subset K of C ,

$$\lim_{n \rightarrow \infty} \sup \left\{ \|T_i^{n+1}x - T_i^n x\| : x \in K \right\}, i = 1, \dots, N,$$

then we obtain the strong convergence of a simpler method than the corresponding ones in Cho, Qin, and Kang [6] and Chang, Kim, and Wang [5].

For the case $N = 1$, Theorem 1 gives the following monotone hybrid method, which modifies the corresponding algorithms in Kim and Xu [17], as well as Kim and Takahashi (Theorems 3.1, 3.7, 4.1 [16]).

Corollary 1 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T : C \rightarrow C$ be an **asymptotically quasi ϕ -nonexpansive** mapping with a sequence $\{k_n\} \subset [1, +\infty)$, $k_n \rightarrow 1$. Moreover, suppose that the mapping T is uniformly L -Lipschitz continuous and the set $F(T)$ is nonempty and bounded in C . Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}x_0 &\in C, C_0 := C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ C_{n+1} &:= \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \varepsilon_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0,\end{aligned}$$

where $\varepsilon_n = (k_n - 1)(\omega + \|x_n\|)^2$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$.

Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_{F(T)} x_0$.

Next, we consider a modified version of the algorithm proposed in Theorem 1.

Theorem 2 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of **asymptotically quasi ϕ -nonexpansive** mappings with a sequence $\{k_n\} \subset [1, +\infty)$, $k_n \rightarrow 1$. Moreover, suppose for each $i \geq 1$, the mapping T_i is uniformly L_i -Lipschitz continuous and the set $F = \bigcap_{i=1}^N F(T_i)$ is nonempty and bounded in C . Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}x_0 &\in C, C_0 := C, \\ y_n^i &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_i^n x_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \quad \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &:= \{v \in C_n : \phi(v, \bar{y}_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n) \phi(v, x_n) + \varepsilon_n\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0,\end{aligned}$$

where $\varepsilon_n = (k_n - 1)(\omega + \|x_n\|)^2$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_0$.

Proof Following five steps in the proof of Theorem 1, we can show that:

(i) C_n and F are closed and convex subset of C for all $n \geq 0$. Therefore, $\Pi_{C_n} x_0, n \geq 0$ and $\Pi_F x_0$ are well-defined.

(ii) $F \subset C_n$ for all $n \geq 0$.

Suppose $F \subset C_n$ for some $n \geq 0$ ($F \subset C_0 = C$). For each $u \in F$, using the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned} \phi(u, \bar{y}_n) &= \|u\|^2 - 2\langle u, J\bar{y}_n \rangle + \|\bar{y}_n\|^2 \\ &\leq \alpha_n \phi(u, x_0) + k_n(1 - \alpha_n) \phi(u, x_n) \\ &\leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, x_n) + (k_n - 1)(1 - \alpha_n) \phi(u, x_n) \\ &\leq \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, x_n) + (k_n - 1)(\omega + \|x_n\|)^2 \\ &= \alpha_n \phi(u, x_0) + (1 - \alpha_n) \phi(u, x_n) + \varepsilon_n. \end{aligned}$$

This implies that $u \in C_{n+1}$. Hence $F \subset C_{n+1}$. By induction, we obtain $F \subset C_n$ for all $n \geq 0$.

(iii) The sequence $\{x_n\}$ converges strongly to some point $p \in C$ as $n \rightarrow \infty$.

For each $u \in F \subset C_n$, using Lemma 1 and taking into account that $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(u, x_0) - \phi(u, x_n) \leq \phi(u, x_0).$$

Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. From (2), $\{x_n\}$ is also bounded. Since $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$ for all $n \geq 0$, by Lemma 1 we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) - \phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0).$$

Thus, the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing, hence it has a finite limit as $n \rightarrow \infty$. Moreover, for all $m \geq n$, we also have $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$. From $x_n = \Pi_{C_n} x_0$ and Lemma 1, we obtain

$$\phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0) \rightarrow 0 \quad (12)$$

as $m, n \rightarrow \infty$. Lemma 2 yields $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in C . Since E is Banach space and C is closed, $x_n \rightarrow p \in C$ as $n \rightarrow \infty$.

(iv) $p \in F$.

In view of $x_{n+1} \in C_{n+1}$ and by the construction of C_{n+1} , we get

$$\phi(x_{n+1}, \bar{y}_n) \leq \alpha_n \phi(x_{n+1}, x_0) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \varepsilon_n. \quad (13)$$

Using $\lim_{n \rightarrow \infty} \alpha_n = 0$, relations (12), (13), and noting that $\varepsilon_n \rightarrow 0$, we find $\phi(x_{n+1}, \bar{y}_n) \rightarrow 0$ as $n \rightarrow \infty$. This together with Lemma 2 imply that $\|x_{n+1} - \bar{y}_n\| \rightarrow 0$. Therefore, $\bar{y}_n \rightarrow p$ and $\|x_n - \bar{y}_n\| \rightarrow 0$. Further, by the definition of i_n , we have $\|x_n - y_n^i\| \leq \|x_n - \bar{y}_n\| \rightarrow 0$ as $n \rightarrow \infty$ for all $i = 1, 2, \dots, N$, hence, from $x_n \rightarrow p$, we obtain

$$\lim_{n \rightarrow \infty} y_n^i = p, \quad i = 1, 2, \dots, N. \quad (14)$$

Taking into account the relation $y_n^i = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_i^n x_n)$, we obtain

$$\|Jy_n^i - JT_i^n x_n\| = \alpha_n \|Jx_0 - JT_i^n x_n\|. \quad (15)$$

Observing that $\{x_n\}$ is bounded, T_i is uniformly L_i - Lipschitz continuous, and the solution set F is not empty, we have $\|Jx_0 - JT_i^n x_n\| \leq \|Jx_0\| + \|JT_i^n x_n\| = \|x_0\| + \|T_i^n x_n\| \leq \|x_0\| + \|T_i^n x_n - T_i^n \xi\| + \|\xi\| \leq \|x_0\| + L_i \|x_n - \xi\| + \|\xi\|$, where $\xi \in F$ is an arbitrary fixed element. The last inequality proves the boundedness of the sequence $\{\|Jx_0 - JT_i^n x_n\|\}$. Using $\lim_{n \rightarrow \infty} \alpha_n = 0$ from (15), we find

$$\lim_{n \rightarrow \infty} \|Jy_n^i - JT_i^n x_n\| = 0.$$

Since $J^{-1} : E^* \rightarrow E$ is uniformly continuous on each bounded subset of E^* , the last relation implies $\lim_{n \rightarrow \infty} \|y_n^i - T_i^n x_n\| = 0$. Hence, from (14) we obtain

$$\lim_{n \rightarrow \infty} T_i^n x_n = p, \quad i = 1, \dots, N.$$

Finally, a similar argument as in Step 5 of Theorem 1 leads to the conclusion that $p \in F$ and $p = x^\dagger = \Pi_F x_0$. The proof of Theorem 2 is complete.

Remark 2 Theorem 2 is an extended version of Theorem 3.1 in [6] and Corollary 2.5 in [7] for a family of **asymptotically quasi- ϕ -nonexpansive** mappings. It also simplifies some previous results of Chang and Yan (Theorem 2.1 [7]) and Cho, Qin, and Kang (Theorem 3.5 [6]). In the case $N = 1$, our method modifies the algorithm of Kim and Takahashi [16].

In the next theorem, we show that for quasi ϕ - nonexpansive mappings $\{T_i\}_{i=1}^N$, the assumptions on their uniform Lipschitz continuity, as well as the boundedness of the set of common fixed points $F = \bigcap_{i=1}^N F(T_i)$ are redundant.

Theorem 3 *Let E be a real uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E , and $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of closed and quasi ϕ - nonexpansive mappings. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in C, C_0 := C, \\ y_n^i &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &:= \{v \in C_n : \phi(v, \bar{y}_n) \leq \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_0$.

Proof Since $\{T_i\}_{i=1}^N : C \rightarrow C$ are quasi ϕ - nonexpansive mappings, for each $i = 1, \dots, N$, we have

$$\phi(p, T_i x) \leq \phi(p, x), \forall p \in F(T_i), x \in C.$$

This implies that $\{T_i\}_{i=1}^N$ are **asymptotically quasi ϕ -nonexpansive** mappings with $k_n = 1, n \geq 1$. **Putting $\varepsilon_n = 0$** and arguing similarly as in the proof of Theorem 1, we get $F \subset C_n$. Using Lemma 1 and the fact that $x_n = \Pi_{C_n} x_0$, we have $\phi(x_n, x_0) \leq \phi(p, x_0)$ for each $p \in F$. Hence, the set $\{\phi(x_n, x_0)\}$ is bounded. This together with inequality (2) implies that $\{x_n\}$ is bounded. Repeating the proof of the relations (3), (8), we obtain

$$\lim_{n \rightarrow \infty} x_n = p, \quad (16)$$

$$\lim_{n \rightarrow \infty} y_n^i = p, i = 1, 2, \dots, N. \quad (17)$$

From the equality $y_n^i = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n)$ we have

$$\|Jy_n^i - JT_i x_n\| = \alpha_n \|Jx_n - JT_i x_n\|.$$

Observing that $\{x_n\} \subset C$ is bounded, from the definition of quasi ϕ - nonexpansive mapping T_i , we get $\phi(p, T_i x_n) \leq \phi(p, x_n)$ for each $p \in F$. Estimate (2) ensures that $\{T_i x_n\}$ is bounded for each $i = 1, \dots, N$. Therefore, $\|Jx_n - JT_i x_n\| \leq \|x_n\| + \|T_i x_n\|$. The last inequality implies that the sequence $\{\|Jx_n - JT_i x_n\|\}$ is bounded. Using $\lim_{n \rightarrow \infty} \alpha_n = 0$ we obtain

$$\lim_{n \rightarrow \infty} \|Jy_n^i - JT_i x_n\| = 0. \quad (18)$$

From (17), (18), by the same way as in the proof of (10), we get

$$\lim_{n \rightarrow \infty} T_i x_n = p, i = 1, 2, \dots, N. \quad (19)$$

By (16), (19) and the closedness of T_i , we obtain $p \in F = \bigcap_{i=1}^N F(T_i)$. Finally, arguing as in Step 5 of the proof of Theorem 1, we can show that $p = x^\dagger$. Thus, the proof of Theorem 3 is complete.

By the same method we can prove the following result.

Theorem 4 *Let E be a real uniformly smooth and uniformly convex Banach space, C be a nonempty closed convex subset of E , and $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of closed and quasi ϕ - nonexpansive mappings. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in C, C_0 := C, \\ y_n^i &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_i x_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &:= \{v \in C_n : \phi(v, \bar{y}_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n) \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_0$.

Remark 3 Theorem 3 modifies Theorem 3.1 [27], Theorem 3.1 [34] and the algorithm in Theorem 3.2 [15]. On the other hand, the method in Theorem 4 simplifies the corresponding one in Theorem 3.3 [27]. It generalizes and improves Theorem 3.2 [26], Theorem 3.3 [5], and Theorem 3.1 in [23].

The following result can be obtained from Theorem 3 immediately.

Corollary 2 *Let E be a real uniformly smooth and uniformly convex Banach space, and C a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of closed relatively nonexpansive mappings. Suppose that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in C, C_0 = C, \\ y_n^i &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &= \{v \in C_n : \phi(v, \bar{y}_n) \leq \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_0$.

Corollary 3 *Let E be a real uniformly smooth and uniformly convex Banach space. Let $\{A_i\}_{i=1}^N : E \rightarrow E^*$ be a finite family of maximal monotone mappings with $D(A_i) = E$ for all $i = 1, \dots, N$. Suppose that the solution set S of the system of operator equations $A_i(x) = 0, i = 1, \dots, N$ is nonempty. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_0 &\in E, C_0 = E, \\ y_n^i &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)J(J + r_i A_i)^{-1} Jx_n), i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{\|y_n^i - x_n\|\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &= \{v \in C_n : \phi(v, \bar{y}_n) \leq \phi(v, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

where $\{r_i\}_{i=1}^N$ are given positive numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_S x_0$.

Proof Let $C = D(A_i) = E$ and $T_i = (J + r_i A_i)^{-1} J : C \rightarrow C$. By Lemmas 5 and 4, the mappings $T_i, i = 1, \dots, N$, are closed and quasi ϕ -nonexpansive. Moreover, $F = \bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N A_i^{-1}(0) = S \neq \emptyset$. Thus, Theorem 3 ensures the conclusion of Corollary 3.10.

3.2 Sequential hybrid methods

Now, we consider a sequential method for finding a common fixed point of a finite family of **asymptotically quasi ϕ -nonexpansive** mappings.

Theorem 5 Let C be a nonempty closed convex subset of a real uniformly smooth and uniformly convex Banach space E , and $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of asymptotically quasi ϕ -nonexpansive mappings with $\{k_n\} \subset [1, +\infty)$, $k_n \rightarrow 1$. Suppose $\{T_i\}_{i=1}^N$ are uniformly L -Lipschitz continuous and the set $F = \bigcap_{i=1}^N F(T_i)$ is unempty and bounded in C , i.e., $F \subset \Omega := \{u \in C : \|u\| \leq \omega\}$ for some positive ω . Let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_1 &\in C_1 = Q_1 := C, \\ y_n &= J^{-1} \left(\alpha_n Jx_1 + (1 - \alpha_n) JT_{j_n}^{p_n} x_n \right), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n) \phi(v, x_n) + \varepsilon_n\}, \\ Q_n &= \{v \in Q_{n-1} : \langle Jx_1 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_1, n \geq 1, \end{aligned}$$

where $n = (p_n - 1)N + j_n$, $j_n \in \{1, 2, \dots, N\}$, $p_n \in \{1, 2, \dots\}$, $\varepsilon_n = (k_{p_n} - 1)(\omega + \|x_n\|)^2$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_1$.

For the proof of Theorem 5 we need the following result.

Lemma 6 Assume that all conditions of Theorem 5 holds. Moreover,

$$\lim_{n \rightarrow \infty} \|x_n - T_{j_n}^{p_n} x_n\| = 0, \lim_{n \rightarrow \infty} \|x_n - x_{n+l}\| = 0$$

for all $l \in \{1, 2, \dots, N\}$. Then

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0, \quad l = 1, \dots, N.$$

Proof For each $n > N$, we have $n = (p_n - 1)N + j_n$. Hence $n - N = ((p_n - 1) - 1)N + j_n = (p_{n-N} - 1)N + j_{n-N}$. So

$$p_n - 1 = p_{n-N}, j_n = j_{n-N}.$$

We have

$$\begin{aligned} \|x_n - T_{j_n} x_n\| &\leq \|x_n - T_{j_n}^{p_n} x_n\| + \|T_{j_n}^{p_n} x_n - T_{j_n} x_n\| \\ &\leq \|x_n - T_{j_n}^{p_n} x_n\| + L \|T_{j_n}^{p_n-1} x_n - x_n\| \\ &\leq \|x_n - T_{j_n}^{p_n} x_n\| + L \|T_{j_n}^{p_n-1} x_n - T_{j_{n-N}}^{p_{n-N}-1} x_{n-N}\| \\ &\quad + L \|T_{j_{n-N}}^{p_{n-N}-1} x_{n-N} - x_{n-N}\| + L \|x_{n-N} - x_n\| \\ &= \|x_n - T_{j_n}^{p_n} x_n\| + L \|T_{j_{n-N}}^{p_{n-N}} x_{n-N} - x_{n-N}\| \\ &\quad + (L^2 + L) \|x_{n-N} - x_n\|. \end{aligned}$$

This together with the hypotheses of Lemma 6 implies

$$\lim_{n \rightarrow \infty} \|x_n - T_{j_n} x_n\| = 0.$$

For each $l \in \{1, 2, \dots, N\}$ we have

$$\begin{aligned} \|x_n - T_{j_{n+l}}x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{j_{n+l}}x_{n+l}\| + \|T_{j_{n+l}}x_{n+l} - T_{j_{n+l}}x_n\| \\ &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{j_{n+l}}x_{n+l}\| + L\|x_{n+l} - x_n\| \\ &= (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{j_{n+l}}x_{n+l}\|. \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - T_{j_{n+l}}x_n\| = 0$ for all $l \in \{1, 2, \dots, N\}$; therefore,

$$\forall \varepsilon > 0, \exists n_0 : \forall n \geq n_0 \forall l = 1, \dots, N, \|x_n - T_{j_{n+l}}x_n\| < \varepsilon.$$

On the other hand, for any fixed $n \geq 0$ and $i = 1, \dots, N$, we can find $l \in \{1, \dots, N\}$, such that $i = j_{n+l}$. Thus, $\|x_n - T_i x_n\| \leq \sup_{l \in \{1, \dots, N\}} \|x_n - T_{j_{n+l}}x_n\| < \varepsilon$ for all $n \geq n_0$, which means that $\lim_{n \rightarrow \infty} \|x_n - T_i x_n\| = 0, i = 1, \dots, N$. The proof of Lemma 6 is complete.

Proof of Theorem 5. The proof will be divided into five steps.

Step 1. The sets F, C_n, Q_n are closed and convex for all $n \geq 1$.

Indeed, from the uniform L -Lipschitz continuity of T_i , we see that T_i is closed. By Lemma 3, $F(T_i)$ is closed and convex subset of C for all $i = 1, \dots, N$. Hence, $F = \bigcap_{i=1}^N F(T_i)$ is closed and convex. Further, C_n and Q_n are closed for all $n \geq 1$ by the definition. From the inequality $\phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n) \phi(v, x_n) + \varepsilon_n$, we obtain

$$2\langle v, Jx_n \rangle + 2\alpha_n \langle v, Jx_1 - Jy_n - Jx_n \rangle \leq \alpha_n \|x_1\|^2 + (1 - \alpha_n) \|x_n\|^2 - \|y_n\|^2 + \varepsilon_n,$$

which implies the convexity of C_n for all $n \geq 1$. Further, $Q_1 = C$ is convex. If Q_n is convex for some $n \geq 1$, then Q_{n+1} is also convex by the definition. So, Q_n is convex for all $n \geq 1$.

Step 2. $F \subset C_n \cap Q_n$ for all $n \geq 1$.

For each $u \in F$, we have

$$\begin{aligned} \phi(u, y_n) &= \|u\|^2 - 2\langle u, Jy_n \rangle + \|y_n\|^2 \\ &= \|u\|^2 - 2\alpha_n \langle u, Jx_1 \rangle - 2(1 - \alpha_n) \langle u, JT_{j_n}^{p_n} x_n \rangle \\ &\quad + \left\| \alpha_n Jx_1 + (1 - \alpha_n) JT_{j_n}^{p_n} x_n \right\|^2 \\ &\leq \|u\|^2 - 2\alpha_n \langle u, Jx_1 \rangle - 2(1 - \alpha_n) \langle u, JT_{j_n}^{p_n} x_n \rangle \\ &\quad + \alpha_n \|x_1\|^2 + (1 - \alpha_n) \left\| T_{j_n}^{p_n} x_n \right\|^2 \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, T_{j_n}^{p_n} x_n) \\ &\leq \alpha_n \phi(u, x_1) + k_{p_n} (1 - \alpha_n) \phi(u, x_n) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + (k_{p_n} - 1) (1 - \alpha_n) \phi(u, x_n) \\ &\leq \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + (k_{p_n} - 1) (\omega + \|x_n\|^2) \\ &= \alpha_n \phi(u, x_1) + (1 - \alpha_n) \phi(u, x_n) + \varepsilon_n. \end{aligned}$$

This implies that $u \in C_n$. Hence $F \subset C_n$ for all $n \geq 1$. We also have $F \subset Q_1 = C$. Suppose that $F \subset Q_n$ for some $n \geq 1$. From $x_{n+1} = \Pi_{C_n \cap Q_n} x_1$ and Lemma 1, it follows that $\langle Jx_1 - Jx_{n+1}, x_{n+1} - z \rangle \geq 0$ for all $z \in C_n \cap Q_n$. Since $F \subset C_n \cap Q_n$, we have

$$\langle Jx_1 - Jx_{n+1}, x_{n+1} - z \rangle \geq 0$$

for all $z \in F$. Hence, from the definition of Q_{n+1} , we obtain $F \subset Q_{n+1}$. By the induction, $F \subset Q_n$ for all $n \geq 1$.

Step 3. $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ for all $l = 1, \dots, N$.

Since $x_n = \Pi_{Q_n} x_1$, $F \subset Q_n$, by Lemma 1, we have $\phi(x_n, x_1) \leq \phi(p, x_1) - \phi(x_n, p) \leq \phi(p, x_1)$ for each $p \in F$. Hence, the sequence $\{\phi(x_n, x_1)\}$ and $\{x_n\}$ are bounded. Moreover, from $x_{n+1} = \Pi_{C_n \cap Q_n} x_1 \in Q_n$, $x_n = \Pi_{Q_n} x_1$ and Lemma 1, it follows that $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. Thus, the sequence $\{\phi(x_n, x_1)\}$ is nondecreasing and the limit of the sequence $\{\phi(x_n, x_1)\}$ exists. This together with $\phi(x_{n+1}, x_n) \leq \phi(x_n, x_1) + \phi(x_{n+1}, x_1)$, implies that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \quad (20)$$

Since $\{x_n\}$ is bounded, there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \geq 1$. Using the boundedness of F and estimate (2), we get

$$\varepsilon_n = (k_{p_n} - 1)(\omega + \|x_n\|)^2 \leq (k_{p_n} - 1)(\omega + M)^2 \rightarrow 0 \quad (n \rightarrow \infty). \quad (21)$$

Taking into account $x_{n+1} = \Pi_{C_n \cap Q_n} x_1 \in C_n$, and using the relations (20), (21), and $\lim_{n \rightarrow \infty} \alpha_n = 0$, from the definition of C_n we find

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n) \phi(x_{n+1}, x_n) + \varepsilon_n \rightarrow 0 \quad (n \rightarrow \infty).$$

Lemma 2 gives

$$\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

and

$$\lim_{n \rightarrow \infty} \|x_{n+l} - x_n\| = 0 \quad (22)$$

for all $l \in \{1, 2, \dots, N\}$. Note that from $y_n = J^{-1} \left(\alpha_n Jx_1 + (1 - \alpha_n) JT_{j_n}^{p_n} x_n \right)$, we have

$$\|Jy_n - JT_{j_n}^{p_n} x_n\| = \alpha_n \|Jx_1 - JT_{j_n}^{p_n} x_n\|. \quad (23)$$

Observing that $\{x_n\}$ is bounded, T_{j_n} is uniformly L -Lipschitz continuous and the solution set F is not empty, we have $\|Jx_1 - JT_{j_n}^{p_n} x_n\| \leq \|Jx_1\| + \|JT_{j_n}^{p_n} x_n\| = \|x_1\| + \|T_{j_n}^{p_n} x_n\| \leq \|x_1\| + \|T_{j_n}^{p_n} x_n - T_{j_n}^{p_n} \xi\| + \|\xi\| \leq \|x_1\| + L\|x_n - \xi\| + \|\xi\|$, where $\xi \in F$ is an arbitrary fixed element. The last inequality proves the boundedness of the sequence $\{\|Jx_1 - JT_{j_n}^{p_n} x_n\|\}$. Using $\lim_{n \rightarrow \infty} \alpha_n = 0$, from (23), we find

$$\lim_{n \rightarrow \infty} \|Jy_n - JT_{j_n}^{p_n} x_n\| = 0.$$

Since $J^{-1} : E^* \rightarrow E$ is uniformly continuous on each bounded set, we get

$$\lim_{n \rightarrow \infty} \|y_n - T_{j_n}^{p_n} x_n\| = 0.$$

This together with $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ implies that

$$\lim_{n \rightarrow \infty} \|x_n - T_{j_n}^{p_n} x_n\| = 0. \quad (24)$$

From (22), (24) and Lemma 6, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad (25)$$

for all $l \in \{1, 2, \dots, N\}$.

Step 4. $\lim_{n \rightarrow \infty} x_n = p \in F$.

Indeed, note that the limit of the sequence $\{\phi(x_n, x_1)\}$ exists. By the construction of Q_n , we have $Q_m \subset Q_n$ for all $m \geq n$. Moreover, $x_n = \Pi_{Q_n} x_1$ and $x_m \in Q_m \subset Q_n$. These together with Lemma 1 imply that $\phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \rightarrow 0$ as $m, n \rightarrow \infty$. By Lemma 2, we get

$\lim_{m, n \rightarrow \infty} \|x_m - x_n\| = 0$. Hence, $\{x_n\}$ is a Cauchy sequence. Since C is a closed and convex subset of the Banach space E , the sequence $\{x_n\}$ converges strongly to $p \in C$. Since T_l is L -Lipschitz continuous mapping, it is continuous for all $l \in \{1, 2, \dots, N\}$. Hence

$$\|p - T_l p\| = \lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0, \forall l \in \{1, 2, \dots, N\}.$$

This implies that $p \in F$.

Step 5. $p = \Pi_F x_1$.

From $x^\dagger := \Pi_F x_1 \in F \subset C_n \cap Q_n$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_1$, we have $\phi(x_{n+1}, x_1) \leq \phi(x^\dagger, x_1)$. Hence

$$\phi(p, x_1) = \lim_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(x^\dagger, x_1).$$

Therefore, $p = x^\dagger$. The proof of Theorem 5 is complete.

For a finite family of closed and quasi ϕ -nonexpansive mappings, the assumption on the boundedness of $F = \bigcap_{i=1}^N F(T_i)$ is redundant.

Theorem 6 *Let E be a real uniformly smooth and uniformly convex Banach space, and C a nonempty closed convex subset of E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of closed and quasi ϕ -nonexpansive mappings. Suppose $\{T_i\}_{i=1}^N$ are L -Lipschitz continuous and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_1 &\in C_1 = Q_1 := C, \\ y_n &= J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_{j_n}x_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in Q_{n-1} : \langle Jx_1 - Jx_n, x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_1, n \geq 1, \end{aligned}$$

where $n = (p_n - 1)N + j_n$, $j_n \in \{1, 2, \dots, N\}$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_1$.

Proof By our assumption, $\{T_i\}_{i=1}^N$ is a finite family of closed and **asymptotically quasi ϕ -nonexpansive** mappings with $k_n = 1$ for all $n \geq 0$. Putting $\varepsilon_n = 0$ and arguing similarly as in the proofs of Theorem 5 and Lemma 6, we obtain $\lim_{n \rightarrow \infty} \|x_n - T_{j_n} x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ for all $l \in \{1, 2, \dots, N\}$. Now repeating Steps 4 and 5 of the proof of Theorem 5, we come to the conclusion of Theorem 6.

Remark 4 One can establish the convergence of a monotone hybrid method as in Theorem 5, which modifies Liu's algorithm [20].

Corollary 4 *Let C be a nonempty closed convex subset of a real uniformly smooth and uniformly convex Banach space E . Let $\{T_i\}_{i=1}^N : C \rightarrow C$ be a finite family of closed relatively nonexpansive mappings. Suppose $\{T_i\}_{i=1}^N$ are L -Lipschitz continuous and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_1 &\in C_1 = Q_1 := C, \\ y_n &= J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)JT_{j_n}x_n), \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in Q_{n-1} : \langle Jx_1 - Jx_n; x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_1, n \geq 1, \end{aligned}$$

where $n = (p_n - 1)N + j_n$, $j_n \in \{1, 2, \dots, N\}$ and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $x^\dagger := \Pi_F x_1$.

Corollary 5 *Let E be a real uniformly smooth and smooth convex Banach space. Let $\{A_i\}_{i=1}^N : E \rightarrow E^*$ be a finite family of maximal monotone mappings with $D(A_i) = E$ for all $i = 1, \dots, N$. Suppose that the solution set S of the system of operator equations $A_i(x) = 0, i = 1, \dots, N$ is nonempty. Let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned} x_1 &\in E, C_1 = E, \\ y_n &= J^{-1}(\alpha_n Jx_1 + (1 - \alpha_n)J(J + r_{j_n}A_{j_n})^{-1}Jx_n), i = 1, 2, \dots, N, \\ C_n &= \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, x_1) + (1 - \alpha_n)\phi(v, x_n)\}, \\ Q_n &= \{v \in Q_{n-1} : \langle Jx_1 - Jx_n; x_n - v \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_1, n \geq 1, \end{aligned}$$

where $\{r_i\}_{i=1}^N$ are given positive numbers and $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then $\{x_n\}$ converges strongly to $x^\dagger := \Pi_S x_1$.

We end this paper by considering a numerical example. Suppose we are given two sequences of positive numbers $0 < t_1 < \dots < t_N < 1$ and $s_i \in (1, \frac{1}{1-t_i}]$; $i = 1, \dots, N$. An example of such $\{s_i\}_{i=1}^N$ are $s_i = \sum_{k=0}^{m_i} t_i^k$, where the integers $m_i \geq 1$ for all $i = 1, \dots, N$. Let $E = \mathbb{R}^1$ be a Hilbert space with the standart inner product $\langle x, y \rangle := xy$ and the norm $\|x\| := |x|$ for all $x, y \in E$. In this case the normalized dual mapping $J = I$ and the Lyapunov functional $\phi(x, y) = |x - y|^2$. We define the mappings $T_i : C \rightarrow C, i = 1, \dots, N$, where $C := [0, 1]$, as follows:
 $T_i(x) = 0$, for $x \in [0, t_i]$, and $T_i(x) = s_i(x - t_i)$, if $x \in [t_i, 1]$.

It is easy to verify that $F(T_i) = \{0\}$, $\phi(T_i(x), 0) = |T_i(x)|^2 \leq |x|^2 = \phi(x, 0)$ for every $x \in C$ and $|T_i(1) - T_i(t_i)| = s_i(1 - t_i) > |1 - t_i|$. Hence, the mappings T_i are quasi ϕ -nonexpansive but not nonexpansive.

According to Theorem 3, the iteration sequence $\{x_n\}$ generated by

$$\begin{aligned} x_0 &\in C, C_0 := C, \\ y_n^i &= \alpha_n x_n + (1 - \alpha_n) T_i x_n, i = 1, 2, \dots, N, \\ i_n &= \arg \max_{1 \leq i \leq N} \{|y_n^i - x_n|\}, \bar{y}_n := y_n^{i_n}, \\ C_{n+1} &:= \{v \in C_n : |v - \bar{y}_n| \leq |v - x_n|\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, n \geq 0, \end{aligned}$$

strongly converges to $x^\dagger := 0$, provided the sequence $\{\alpha_n\}$ is chosen such that $\alpha_n \in [0, 1]$ and $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

Starting from $C_0 = C = [0, 1]$ we have

$$C_1 = \{v \in C_0 : 2(\bar{y}_0 - x_0)(\frac{x_0 + \bar{y}_0}{2} - v) \leq 0\}. \quad (26)$$

Due to the proof of Theorem 3, $F = \{0\} \subset C_1$, hence $(\bar{y}_0 - x_0)(\frac{x_0 + \bar{y}_0}{2}) \leq 0$. Thus, $\bar{y}_0 \leq x_0$. If $\bar{y}_0 = x_0$ then from the definition of i_0 , we find $y_0^i = x_0$ for all $i = 1, \dots, N$. Moreover, since $y_0^i = \alpha_0 x_0 + (1 - \alpha_0) T_i x_0$, we get $x_0 = \alpha_0 x_0 + (1 - \alpha_0) T_i x_0, i = 1, \dots, N$, hence, x_0 is a desired common fixed point and the algorithm finishes at step $n = 0$. Now suppose that $\bar{y}_0 < x_0$. Then (26) implies that $C_1 = [0, \frac{x_0 + \bar{y}_0}{2}]$ and $x_1 = \Pi_{C_1} x_0 = \frac{x_0 + \bar{y}_0}{2}$.

We assume by induction that at the n -th step ($n \geq 1$), either x_{n-1} is a common fixed point of $T_i, i = 1, \dots, N$, and the algorithm finishes at the $(n-1)$ -step, or $C_n = [0, \frac{x_{n-1} + \bar{y}_{n-1}}{2}]$ and $x_n = \Pi_{C_n} x_0 = \frac{x_{n-1} + \bar{y}_{n-1}}{2}$. By the definition of C_{n+1} we have $C_{n+1} = \{v \in C_n : 2(\bar{y}_n - x_n)(\frac{x_n + \bar{y}_n}{2} - v) \leq 0\}$, or equivalently,

$$C_{n+1} = [0, \frac{x_{n-1} + \bar{y}_{n-1}}{2}] \cap \{v \in [0, 1] : 2(\bar{y}_n - x_n)(\frac{x_n + \bar{y}_n}{2} - v) \leq 0\} \quad (27)$$

Since $F = \{0\} \subset C_{n+1}$, we find that $(\bar{y}_n - x_n)(\frac{x_n + \bar{y}_n}{2}) \leq 0$, hence $\bar{y}_n \leq x_n$. If $\bar{y}_n = x_n$ then by the definition of i_n , we get $y_n^i = x_n$ for all $i = 1, \dots, N$. On the other hand, $y_n^i = \alpha_n x_n + (1 - \alpha_n) T_i x_n$, hence, $x_n = \alpha_n x_n + (1 - \alpha_n) T_i x_n$. Thus, x_n is a common fixed point of the family $\{T_i\}_{i=1}^N$ and the algorithm finishes at the n -th step. In the remaining case $\bar{y}_n < x_n$, relation (27) gives

$$C_{n+1} = [0, \frac{x_{n-1} + \bar{y}_{n-1}}{2}] \cap [0, \frac{x_n + \bar{y}_n}{2}]. \quad (28)$$

Noting that $\frac{x_n + \bar{y}_n}{2} < x_n = \frac{x_{n-1} + \bar{y}_{n-1}}{2}$, and using (28) we come to the conclusion that $C_{n+1} = [0, \frac{x_n + \bar{y}_n}{2}]$, and $x_{n+1} = \Pi_{C_{n+1}} x_0 = \frac{x_n + \bar{y}_n}{2}$.

On the other hand, applying Liu's sequential method [20], at the n -th iteration, we need to compute $y_n := \alpha_n x_0 + (1 - \alpha_n) T_{k_n} x_n$, where $k_n = n(\text{mod } N) + 1$. Observing that $0 \leq T_{k_n} x_n \leq x_n \leq 1$, we have if $x_n = T_{k_n} x_n$ then x_n is a fixed point of T_{k_n} , which

is also a common fixed point of the family $\{T_i\}_{i=1}^N$. Otherwise, we get $T_{k_n}x_n < x_n$, which leads to the formula

$$x_{n+1} = \min\left\{x_n, \frac{\alpha_n x_0^2 + (1 - \alpha_n)x_n^2 - y_n^2}{2(\alpha_n x_0 + (1 - \alpha_n)x_n - y_n)}\right\}.$$

The numerical experiment is performed on a LINUX cluster 1350 with 8 computing nodes. Each node contains two Intel Xeon dual core 3.2 GHz, 2GBRam. All the programs are written in C.

For given tolerances we compare execution time of the parallel hybrid method (PHM) and Liu's sequential method (LSM) [20]. From Tables 1-3, we see that within a given tolerance, the sequential method is more time consuming than the parallel one, in both parallel and sequential mode. Further, whenever the tolerance is small, the sequential method converges very slowly or practically diverges.

We use the following notations:

PHM	The parallel hybrid method
LSM	Liu's sequential method [20]
N	Number of quasi ϕ -nonexpansive mappings
TOL	Tolerance $\ x_k - x^*\ $
very slow conv.	Convergence is very slow or divergence
T_p	Time for PHM's execution in parallel mode (2CPUs - in seconds)
T_s	Time for PHM's execution in sequential mode (in seconds)
T_L	Time for LSM's execution (in seconds).

We perform experiments with $N = 5 \times 10^6$, $t_i = \frac{i}{N+1}$, $s_i = 1 + t_i$, $i = 1, \dots, N$. Within the tolerance $TOL = 10^{-4}$, for $\alpha_n = 1/n$ and $\alpha_n = 10^{-n}$, the computing

Table 1 Experiment with $\alpha_n = 1/n$.

TOL	PHM		LSM
	T_p	T_s	T_L
10^{-5}	1.06	1.90	Very slow conv.
10^{-6}	1.26	2.10	Very slow conv.
10^{-8}	1.47	2.74	Very slow conv.

Table 2 Experiment with $\alpha_n = \frac{1}{\log n + 2}$.

TOL	PHM		LSM
	T_p	T_s	T_L
10^{-5}	1.27	2.52	Very slow conv.
10^{-6}	1.48	2.95	Very slow conv.
10^{-8}	1.89	3.58	Very slow conv.

Table 3 Experiment with $\alpha_n = 10^{-n}$.

TOL	PHM		LSM
	T_p	T_s	T_L
10^{-5}	0.84	1.68	Very slow conv.
10^{-6}	1.05	1.90	Very slow conv.
10^{-8}	1.26	2.31	Very slow conv.

times of Liu's method are 30.89 sec. and 26.57 sec., respectively. Moreover, for $\alpha_n = 1/(\log n + 2)$, after 287.25 sec., Liu's method gives an approximate solution $\tilde{x} = 0.327$, which is very far from the exact solution $x^* = 0$. When $TOL = 10^{-k}$, $k = 5, 6, 8$, Liu's method is practically divergent.

Tables 1-3 give the execution times of the parallel hybrid method in parallel mode (T_p) and sequential mode (T_s) within the given tolerances TOL for different choices of α_n . The maximal speed up of the parallel hybrid method is $S_p := T_s/T_p \approx 2.0$, hence, the efficiency of the parallel computation by using two processors is $E_p := S_p/2 \approx 1.0$.

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